# Weighted Model Counting in FO$^2$ with Cardinality Constraints and Counting Quantifiers


Sagar Malhotra* [1,2] Luciano Serafini† [1]

[1]Fondazione Bruno Kessler, Italy [2]University of Trento, Italy


## Weighted First Order Model Counting

$$\mathrm{FOMC}(\Phi, n) = \sum_{\omega \in \Omega} \mathbb{1}(\omega \models \Phi)$$

$$\mathrm{WFOMC}(\Phi, n) = \sum_{\omega \in \Omega} \mathbb{1}(\omega \models \Phi) \times w(\omega)$$

Example:

$$\Phi = \forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}$$

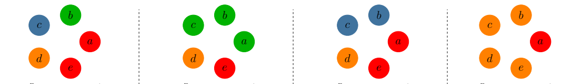
$$\mathrm{FOMC}(\Phi, n)?$$

## Unary and Binary Properties in FO$^2$

Let us have a FOL language with a unary predicate $\mathtt{A}$ and a binary predicate $\mathtt{R}$. Then for any domain constant $\mathtt{c}$ exactly one of the following **unary** property is true:

$$\mathtt{Ac} \land \mathtt{Rcc} \quad | \quad \mathtt{Ac} \land \neg\mathtt{Rcc} \quad | \quad \neg\mathtt{Ac} \land \mathtt{Rcc} \quad | \quad \neg\mathtt{Ac} \land \neg\mathtt{Rcc} \tag{1}$$

For 5 domain elements some examples of **unary** configurations are given as follows:



$$\binom{5}{2,1,1,1} = \frac{5!}{2!1!1!1!} \quad k_1 = 2 \quad k_2 = 1 \quad k_3 = 1 \quad k_4 = 1$$

$$\binom{5}{1,3,0,1} = \frac{5!}{1!3!0!1!} \quad k_1 = 1 \quad k_2 = 3 \quad k_3 = 0 \quad k_4 = 1$$

$$\binom{5}{2,0,2,1} = \frac{5!}{2!0!2!1!} \quad k_1 = 2 \quad k_2 = 0 \quad k_3 = 2 \quad k_4 = 1$$

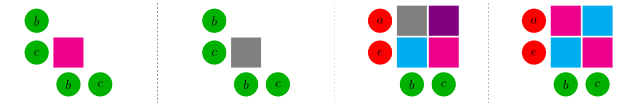
$$\binom{5}{1,0,0,4} = \frac{5!}{1!0!0!4!} \quad k_1 = 2 \quad k_2 = 1 \quad k_3 = 1 \quad k_4 = 1$$

*In general, for a language with $u$ **unary** properties over $n$ domain elements, we have $\binom{n}{\vec{k}} = \frac{n!}{\prod_i k_i!}$ ways such that $k_i$ constants realize the $i^{th}$ property, where $\vec{k} = (k_1, ..., k_u)$.*

For any pair of domain constants $(\mathtt{c}, \mathtt{d})$, exactly one of the following **binary** property is true:

$$\mathtt{Rcd} \land \mathtt{Rdc} \quad | \quad \mathtt{Rcd} \land \neg\mathtt{Rdc} \quad | \quad \neg\mathtt{Rcd} \land \mathtt{Rdc} \quad | \quad \neg\mathtt{Rcd} \land \neg\mathtt{Rdc} \tag{2}$$

Given a unary configuration $a$ $e$ $b$ $c$ $d$ of domain elements. Following are some possible realizations of binary properties by the pairs of domain elements.



$$\binom{k_2 \cdot (k_2 - 1)/2}{0,0,1,0} = \frac{(1 \times 1)!}{0!1!0!0!} \quad h_1^{22} = 0 \quad h_2^{22} = 0 \quad h_3^{22} = 1 \quad h_4^{22} = 0$$

$$\binom{k_2 \cdot (k_2 - 1)/2}{0,1,0,0} = \frac{(1 \times 1)!}{0!1!0!0!} \quad h_1^{22} = 0 \quad h_2^{22} = 1 \quad h_3^{22} = 0 \quad h_4^{22} = 0$$

$$\binom{k_1 \cdot k_2}{1,1,1,1} = \frac{(2 \times 2)!}{1!1!1!1!} \quad h_1^{12} = 1 \quad h_2^{12} = 1 \quad h_3^{12} = 1 \quad h_4^{12} = 1$$

$$\binom{k_1 \cdot k_2}{2,0,2,0} = \frac{(2 \times 2)!}{2!0!2!0!} \quad h_1^{12} = 2 \quad h_2^{12} = 0 \quad h_3^{12} = 2 \quad h_4^{12} = 0$$

*In general, for a language with $b$ **binary** properties, given a configuration of unary properties by $\vec{k}$, then for any pair of unary properties $i$ and $j$, we have $\binom{k(i,j)}{h_1^{ij} \dots h_b^{ij}}$ possible ways such that $h_v^{ij}$ pairs of constants realize the $v^{th}$ **binary** property, where*

$$k(i, j) = \begin{cases} k_i \cdot (k_i - 1)/2 & i = j \\ k_i \cdot k_j & i \neq j \end{cases}$$

## $\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}), n)$

Using arguments from the previous section we have that the number of interpretations such that $k_i$ constants (say $\mathtt{c}$) realize the $i^{th}$ **unary** property (denoted by $\mathtt{i(c)}$), and $h_v^{ij}$ pairs of constants $(\mathtt{c}, \mathtt{d})$ such that $\mathtt{i(c)} \land \mathtt{j(d)}$ and the pair $(\mathtt{c}, \mathtt{d})$ realizes the $v^{th}$ **binary** property i.e. $\mathtt{i(c)} \land \mathtt{j(d)} \land \mathtt{v(c,d)}$ is given by :

$$\binom{n}{\vec{k}} \prod_{1 \leq i \leq j \leq u} \binom{k(i,j)}{\vec{h^{ij}}} \tag{3}$$

$\omega \models \forall xy.\Phi(x, y)$ if and only if all the property configurations of each pair of domain constants in $\omega$ is allowed by the formula $\forall xy.\Phi(x, y)$. For example, $\forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}$ does not allow a pair of constants $(\mathtt{c}, \mathtt{d})$ such that $\mathtt{Ac} \land \mathtt{Rcc} \land \neg\mathtt{Ad} \land \neg\mathtt{Rdd} \land \mathtt{Rcd} \land \mathtt{Rdc}$ i.e. the following sub-structure is never allowed:

Hence, we introduce an indicator variable $n_{ijv}$ for each configuration $\mathtt{i(c)} \land \mathtt{j(d)} \land \mathtt{v(c,d)}$ which is 1 if:

$$\mathtt{i(x)} \land \mathtt{j(y)} \land \mathtt{v(x,y)} \models \Phi(\mathtt{x},\mathtt{x}) \land \Phi(\mathtt{x},\mathtt{y}) \land \Phi(\mathtt{y},\mathtt{x}) \land \Phi(\mathtt{y},\mathtt{y})$$

and 0 otherwise.

Hence, given a configuration represented by $\vec{k}$ and $\{\vec{h^{ij}}\}_{ij}$ we have the following possible realizations:

$$F(\vec{k}, \vec{h}, \{n_{ijv}\}) = \binom{n}{\vec{k}} \prod_{1 \leq i \leq j \leq u} \binom{k(i,j)}{\vec{h^{ij}}} \prod_{0 \leq v \leq b} (n_{ijv})^{h_v^{ij}} \tag{4}$$

Hence,

$$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}), n) = \sum_{\vec{k}, \vec{h}} F(\vec{k}, \vec{h}, \{n_{ijv}\}) \tag{5}$$

## Cardinality Constraints

Cardinality Constraints are constraints on the number of times a certain predicate is true in a given FOL interpretation.

Example:

$$\Phi := (\forall \mathtt{xy}.\mathtt{Ax} \land \mathtt{Rxy} \rightarrow \mathtt{Ay}) \land (|\mathtt{A}| = \mathtt{m})$$

*Counting with a Cardinality Constraint $\rho$ can be done by simply allowing cardinality configurations of the properties, which agree with the cardinality constraint.*

$$\mathrm{FOMC}(\Phi \land \rho, n) = \sum_{\rho \models \vec{k}, \vec{h}} F(\vec{k}, \vec{h}, \{n_{ijv}\}) \tag{6}$$

In the above example, we can obtain the cardinality constraint by simply defining

$$\rho := k_1 + k_2 = m$$

## Principle of Inclusion Exclusion

- Let $\Omega$ be a set of objects
- $\mathcal{S} = \{S_1, \dots, S_m\}$ be a set of properties of $\Omega$
- $e_0$ : The count of objects with NONE of the properties in $\mathcal{S}$
- Let $Q \subseteq \mathcal{S}$, then $N_Q$ is the count of objects with AT LEAST the properties in $Q$

We define,

$$s_l = \sum_{|Q| = l} N_Q \tag{7}$$

Then the following relation holds:

$$e_0 = \sum_{l=0}^{m} (-1)^l s_l \tag{8}$$

## Existential Quantifiers (Special Case)

$$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x} \exists \mathtt{y}.\mathtt{Rxy}, n)?$$

$$\Omega = \{\omega : \omega \models \forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y})\} \tag{9}$$

$$S_c = \{\omega : \omega \models \forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{y}.\neg\mathtt{Rcy}\} \tag{10}$$

$$s_l = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \mathtt{Px} \rightarrow \neg\mathtt{Rxy} \land (|\mathtt{P}| = l)) \tag{11}$$

$$e_0 = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x} \exists \mathtt{y}.\mathtt{Rxy}) \tag{12}$$

## Counting Quantifiers (Special Case)

$$\mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.(\mathtt{A(x)} \leftrightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}), n)?$$

STEP 1: FOMC for :

$$\begin{aligned} &\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}) \\ &\land \forall \mathtt{x}.(\mathtt{Bx} \rightarrow \neg\mathtt{Ax}) \end{aligned} \tag{13}$$

which is equal to FOMC for:

$$\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists \mathtt{y}.\mathtt{Rxy}) \tag{14}$$

$$\land \forall \mathtt{x}.(\mathtt{Bx} \rightarrow \neg\mathtt{Ax}) \tag{15}$$

$$\land \forall \mathtt{xy}.\mathtt{Mxy} \leftrightarrow ((\mathtt{Ax} \lor \mathtt{Bx}) \land \mathtt{Rxy}) \tag{16}$$

$$\land |\mathtt{M}| = |\mathtt{A}| + |\mathtt{B}| \tag{17}$$

STEP 2: Inclusion Exclusion:

**KEY IDEA:** Let $S_c = \{\omega : \omega \models \neg\mathtt{Ac} \land \exists^{=1}\mathtt{y}.\mathtt{Rcy}\}$ Clearly, we want the count of models $\omega$ such that $\omega \notin S_c$ for any $c$ i.e.

$$e_0 = \mathrm{FOMC}(\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.(\mathtt{Ax} \leftrightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy}))$$

$$\begin{aligned} s_l = \mathrm{FOMC}(&\forall \mathtt{xy}.\Phi(\mathtt{x}, \mathtt{y}) \land \forall \mathtt{x}.((\mathtt{Ax} \lor \mathtt{Bx}) \rightarrow \exists^{=1}\mathtt{y}.\mathtt{Rxy})) \\ &\land \forall \mathtt{x}.(\mathtt{Ax} \rightarrow \neg\mathtt{Bx}) \land (|\mathtt{B}| = l)) \end{aligned} \tag{18}$$

## Weighted Model Counting

FOMC can be converted to WFOMC by just adding a multiplicative factor $w(\vec{k}, \vec{h})$ to every occurrence of $F(\vec{k}, \vec{h}, \{n_{ijv}\})$ in any counting formula:

$$(\vec{k}, \vec{h}) \mapsto w(\vec{k}, \vec{h}) \in \mathbb{R}^+$$

$w(\vec{k}, \vec{h})$ is a strictly more expressive weight function than symmetric weight functions.


*smalhotra@fbk.eu,†serafini@fbk.eu



To appear in Proceedings of AAAI 2022